Numerical Implementation of BDF2 via Method of Lines for Time Dependent Nonlinear Burgers' Equation

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Abstract – In this paper an efficient unconditionally stable numerical scheme is proposed for solving one dimensional quasi linear Burgers' equation. The proposed scheme comprises of semi discretization via method of lines for the space variable and backward differentiation formula of order two (BDF2) for the time variable. The method of lines reduces the quasi linear partial differential equation in to nonlinear ordinary differential equations at each node point. The resulting nonlinear system is solved by an efficient stiff solver known as BDF2. BDF2 is an implicit solver which leads to nonlinear algebraic system and the resulting nonlinear algebraic system is linearized via Taylor series. This linearization technique is easy to implement and the accuracy of the method will remain unchanged. The linearized system of algebraic equations is solved using MATLAB 8.0. The proposed scheme is implemented on test examples and it has been observed that the numerical solution lies very close to the exact solution. Various numerical experiments have been carried out to demonstrate the performance of the method.

Index Terms – Burgers' equation ; Kinematic viscosity; Method of lines; Backward Differentiation Formula; Taylor series.

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1 INTRODUCTION

In this paper, we consider the quasilinear one dimensional Burgers' equation

$$u_t + uu_x = \nu u_{xx}, x \in [0,1] \text{ and } t \in [0,T] \quad (1.1a)$$

with initial condition

$$u(x,0) = u_0(x), \quad 0 \le x \le 1, \quad (1.1b)$$

and boundary conditions

$$u(0,t) = 0, \quad 0 \le t \le T, \quad (1.1c)$$

$$u(1,t) = 0, \quad 0 \le t \le T. \quad (1.1d)$$

where $v > 0$ is the kinematic viscosity parameter and $u_0(x)$ is given sufficiently smooth function. This equation is known as Burgers' equation which is named after J. M. Burgers [5], [6] due to his enormous contributions. It was introduced by Bateman [2] in 1915. The nonlinear physical phenomena "turbulence" is modelled by this equation. It's structure is similar to Navier-Stoke's equation and hence precise analytic solution for this equation does not exist. Hence several scientists and mathematicians are interested in finding its numerical solution. So far, various numerical methods have been developed such as, finite element method method [15], Adomian's decomposition method [1], Petro-Galerkin method [10], explicit and exact-explicit finite difference method [13], a mixed finite difference and boundary element approach [3], B- spline finite element method [14], Crank-Nicolson scheme [12], Douglas finite difference scheme [16], meshless method of lines [9], pseudospectral method and Darvishi's preconditioning [7] , lattice Boltzmann method [8] and Haar wavelet quasilinearization approach [11] .

In this paper, quasi linear one-dimensional Burgers' equation is solved by Method of lines (MOL) in which the spatial derivatives are approximated by finite differences. The quasi linear partial differential equation gets converted into a nonlinear system of ordinary differential equations in time variable. This system is solved by Backward Differentiation Formula of order two (BDF-2) combined with Taylor series expansion. Taylor series expansion is used for linearization and linear algebraic equations are solved directly thereby increasing the efficiency of the proposed scheme. The proposed method has accuracy of order two in space and time.

2 Numerical Scheme

The Numerical scheme proposed in this paper comprises of Method of Lines (MOL), Backward Differentiation Formula of order two (BDF2) and Linearization technique. MOL is a semi-discretization technique in which discretization is done only along the spatial direction. We divide the spatial direction into N +1 equally spaced points with space interval $\Delta x = 1/N$. Spatial derivatives are approximated using central difference scheme as given below.

$$\frac{\partial u}{\partial x}(x_i,t)=\frac{u_{i+1}(t)-u_{i-1}(t)}{2\Delta x}, i=1,2,\ldots N-1.$$

$$\frac{\partial^2 u}{\partial x^2}(x_i,t)=\frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{(\Delta x)^2}, i=1,2,\ldots N-1.$$

Substituting in Burgers' equation Eq. (1.1), and taking into account that $u_0(t)=0$ and $u_N(t)=0$ we obtain a system of nonlinear ordinary differential equations with initial condition

$$\frac{du_i(t)}{dt}=\frac{\nu}{h^2}\left(u_{i+1}(t)-2u_i(t)+u_{i-1}(t)\right)-\frac{u_i(t)}{2h}\left(u_{i+1}(t)-u_{i-1}(t)\right)$$

$$u_i(0)=u_0(x_i), i=1,2,\ldots N-1$$

where, $u_i(t)=u(x_i,t)$, this system of (N-1) differential equations can be written in matrix form as

$$\frac{dU}{dt}=F(U), \qquad (2.2)$$

$$U(0)=U_0$$

where, $U(t)=[u_1(t),\ldots,u_{N-1}(t)]^T$.

F is a nonlinear function of U with elements f_j given as follows.

$$f_i(u_1,\ldots,u_{N-1},t)=u_{i+1}(\lambda_1-\lambda_2 u_i)-u_i(2\lambda_1-\lambda_2 u_{i-1})+\lambda_1 u_{1-1,} \qquad (2.3)$$

where $i=1,2,\ldots N-1$

$$\lambda_1=\frac{\nu}{(\Delta x)^2}, \lambda_2=\frac{1}{(2\Delta x)}$$

The system (2.2) is a nonlinear system of ordinary differential equations which can be solved by integrating in time variable. Divide the time interval into M+1 equally spaced points with time step $\Delta t = 1/M$. For time integration we use Backward Differentiation Formula of order two given below.

2.0.1 Backward Differentiation Formula of order two (BDF-2)

$$U^{(n+1)}=\frac{4}{3}U^n-\frac{1}{3}U^{n-1}+\frac{2}{3}(\Delta t)F\left(U^{n+1},t^{n+1}\right), n=2,\ldots M \qquad (2.4)$$

the solution at first time level i.e. U^1 is obtained from BDF-1.
Backward Differentiation Formula of order one (BDF-1)

$$U^{n+1}=U^n+(\Delta t)F\left(U^{n+1},t^{n+1}\right), n=0,1,\ldots M-1$$

U^0 is the initial condition and $U^{(n)}=[u_1^{(n)},\ldots u_{n-1}^{(n)}]$.
Since the system (2.3) is nonlinear, it require solving a nonlinear algebraic equation at each time level. This can be avoided by using the linearization technique. Linearize by Taylor series

$$F\left(U^{(n+1)}\right)=F\left(U^n\right)+J_F^{(n)}\left(U^{n+1}-U^{(n)}\right)+0\left(\Delta t^2\right), \qquad (2.5)$$

where

$$J_F^{(n)}=\begin{pmatrix} \left(\frac{\partial f_1}{\partial u_1}\right)^{(n)} & \left(\frac{\partial f_1}{\partial u_2}\right)^{(n)} & \cdots & \left(\frac{\partial f_1}{\partial u_{N-1}}\right)^{(n)} \\ \vdots & & & \\ \left(\frac{\partial f_{N-1}}{\partial u_1}\right)^{(n)} & \left(\frac{\partial f_{N-1}}{\partial u_2}\right)^{(n)} & \cdots & \left(\frac{\partial f_{N-1}}{\partial u_{N-1}}\right)^{(n)} \end{pmatrix}$$

is the Jacobian matrix at the n^{th}time level. Substituting Eq. (2.5) in Eq. (2.4) we get,

$$U^{(n+1)}=\frac{4}{3}U^{(n)}-\frac{1}{3}U^{(n-1)}+\frac{2\Delta t\left[F\left(U^{(n)}\right)+J_F^{(n)}\left(U^{(n+1)}-U^{(n)}\right)\right]}{3}$$

$$\left(I-\frac{2\Delta t}{3}J_F^{(n)}\right)U^{(n+1)}=\left(\frac{4}{3}I-\frac{2\Delta t}{3}J_F^{(n)}\right)U^{(n)}+\frac{2\Delta tF\left(U^{(n)}\right)}{3}-\frac{1}{3}U^{(n-1)}$$

$$U^{(n+1)}=\left(I-\frac{2\Delta t}{3}J_F^{(n)}\right)^{-1}\left(\frac{4}{3}I-\frac{2\Delta t}{3}J_F^{(n)}\right)U^{(n)}+\left(I-\frac{2\Delta t}{3}J_F^{(n)}\right)^{-1}\frac{2\Delta tF\left(U^{(n)}\right)}{3}-\left(I-\frac{2\Delta t}{3}J_F^{(n)}\right)^{-1}\frac{1}{3}U^{(n-1)} \quad (2.6)$$

where $J_F^{(n)}$ is the Jacobian matrix at the n[th] time level. Hence the above scheme is linearized. Unlike Newton's method at each time step we need only to solve linear algebraic equations Eq. (2.6) which take less computation time.

3. NUMERICAL RESULTS AND DISCUSSION

Several test experiments were carried out to show the efficiency and adaptability of the proposed numerical scheme. We have compared the computed solution with exact solution for different values of kinematic viscosity v and at different values of final time.

Test Problem Consider the Burgers' Equation

$$u_t+uu_x=vu_{xx}, x\in[0,1] \text{ and } t\in[0,T] \quad (3.7a)$$

with the initial condition

$$u(x,0)=\sin(\pi x), \quad 0\le x\le 1, \quad (3.7b)$$

and the homogeneous boundary conditions

$$u(0,t)=u(1,t)=0, 0\le t\le T. \quad (3.7c)$$

The exact solution of the problem is

$$u(x,t)=2\pi v\frac{\sum_{n=1}^{\infty}C_n\exp\left(-n^2\pi^2vt\right)n\sin(n\pi x)}{C_0+\sum_{n=1}^{\infty}C_n\exp\left(-n^2\pi^2vt\right)\cos(n\pi x)} \quad (3.8a)$$

where, $C_0=\int_0^1\exp\left\{-\frac{1}{2\pi v}[1-\cos(\pi x)]\right\}dx,$ (3.8b)

$$C_n=2\int_0^1\exp\left\{-\frac{1}{2\pi v}[1-\cos(\pi x)]\right\}\cos(n\pi x)dx, \quad (3.8c)$$

TABLE 1
Comparison of the numerical solution with the exact solution at different space points for test problem at T =0.01 for v = 0.1 and Δt=0.0001.

x	Computed Solution			Exact Solution
	N = 20	N = 40	N = 80	
0.1	0.29729	0.29727	0.29726	0.29726
0.2	0.56784	0.56779	0.56780	0.56777
0.3	0.78667	0.78661	0.78662	0.78659
0.4	0.93251	0.93246	0.93246	0.93244
0.5	0.98974	0.98972	0.98972	0.98971
0.6	0.95034	0.95035	0.95035	0.95035
0.7	0.81554	0.81557	0.81558	0.81558
0.8	0.59673	0.59677	0.59679	0.59678
0.9	0.31519	0.31519	0.31521	0.31520

TABLE 2
Comparison of the numerical solution with the exact solution at different space points for test problem at v = 0.01, Δx = 0.00833 and Δt = 0.001.

x	T	Computed Solution	Exact Solution
0.25	0.1	0.56647	0.56633
	0.3	0.39521	0.39503
	0.5	0.30129	0.30115
0.5	0.1	0.94749	0.94741
	0.3	0.74798	0.74771
	0.5	0.58897	0.58870
0.75	0.1	0.86021	0.86013
	0.3	0.96585	0.96567
	0.5	0.83833	0.83803

Fig a

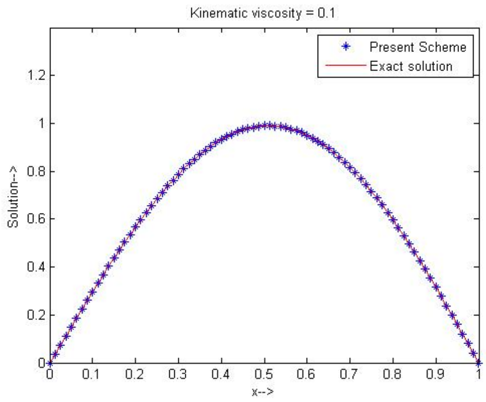

Fig b

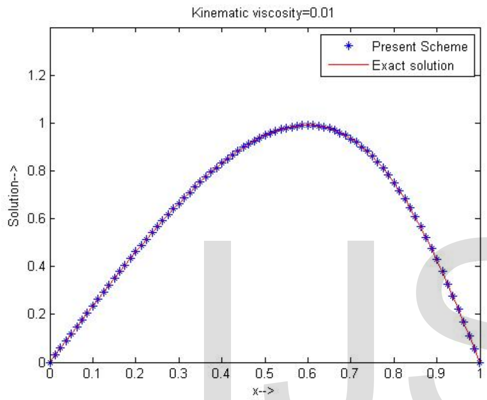

Fig. 1. Numerical solution of test problem for Δx = 0.0125 and different values of v and Δt, (a) v = 0.1, Δt = 0.0001, T = 0.01 (b) v = 0.01, Δt = 0.001, T = 0.1.

Fig a

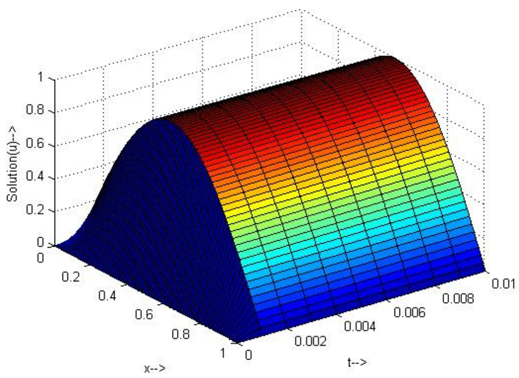

Fig b

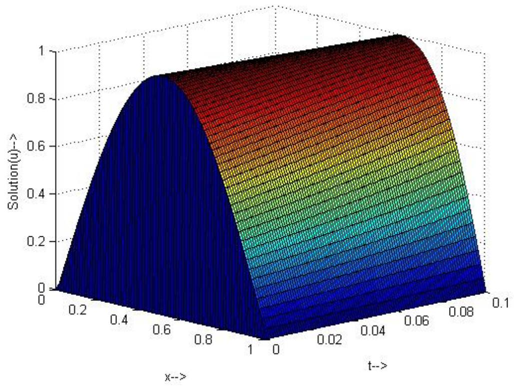

Fig. 2. Physical behaviour of numerical solutions of test problem at different times in 3D for Δx = 0.0125, (a) v = 0.05, Δt = 0.001, T = 0.01 (b) v = 0.0005, Δt = 0.001, T = 0.1.

Fig. a

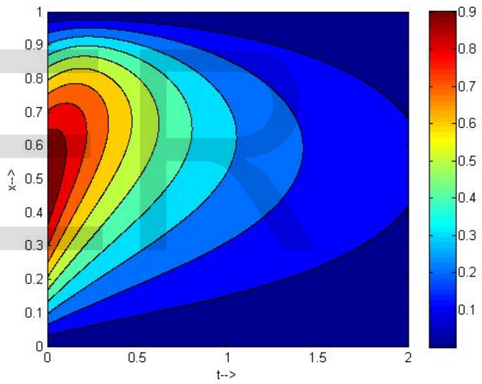

Fig. b

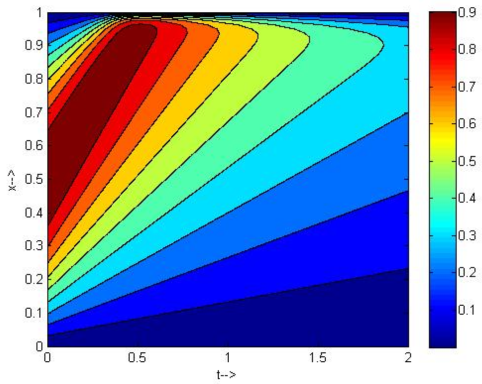

Fig.3. Physical behaviour of numerical solutions of test problem in contour plot for Δx = 0.0125, (a) v = 0.1, Δt = 0.001,T = 2 (b) v = 0.01, Δt = 0.001,T = 2.

The Burgers' equation has been solved both analytically and numerically by the scheme proposed in this paper. Table 1 shows comparison between computed and exact solution with different number of partitions on X-axis for kinematic viscosity, $v = 0.1$. It is clear that as number of partition refines, the approximate solution lies closer to exact, indicating the consistency of the proposed scheme. In Table2, exact and computed solutions are compared at different time levels T = 0.1,0.3,0.5 for kinematic viscosity, $v = 0.01$. Figure 1 shows that numerical solution agrees exactly with analytic solution at each nodal points. The physical behaviour of computed solution is depicted in Figs. 2 and 3 through contour and surface plots for different values of kinematic viscosity $v = 0.1, 0.05, 0.01, 0.005$.

4 Conclusion

In this paper, Burgers' equation has been solved by semi discretization technique and backward differentiation formula of order two (BDF2). This scheme is tested on test example and numerical solution have been compared with exact at different times, for modest values of kinematic viscosity. The numerical results shows excellent agreement with exact which shows the accuracy of the proposed scheme. Linearization technique reduces the computational time as well as cost making the present numerical scheme efficient than the schemes in literature.

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